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LOWER BOUNDS FOR THE LIFE SPAN OF SOLUTIONS OF NONLINEAR WAVE E--ETC(U)
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MRC Technical Summary Report #2393

LOWER BOUNDS FOR THE LIFE SPAN OF SOLUTIONS OF NONLINEAR WAVE EQUATIONS IN THREE DIMENSIONS

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June 1982

(Received April 23, 1982)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triange Park
North Carolina 27709

Office of Naval Research
Arlington, VA 22217

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

The paper deals with strict solutions $u(x,t) = u(x_1,x_2,x_3,t)$ of an equation

$$u_{tt} - \sum_{i,k=1}^{3} a_{ik}(Du)u_{x_i x_k} = 0$$

where Du is the set of 4 first derivatives of u. For given initial values $u(x,0) = \varepsilon F(x)$, $u_t(x,0) = \varepsilon G(x)$ the life span $T(\varepsilon)$ is defined as the supremum of all t to which the local solution can be extended for all x. Blow-up in finite time corresponds to $T(\varepsilon) < \infty$. Examples show that this can occur for arbitrarily small ε. On the other hand $T(\varepsilon)$ must at least be very large for small ε. Assuming that $a_{ik}, F, G \in C^\infty$, that $a_{ik}(0) = \delta_{ik}$, and that F, G have compact support, it is shown that $\lim_{\varepsilon\to 0} \varepsilon^N T(\varepsilon) = \infty$ for every N. This result had been established previously only for $N < 4$.

AMS (MOS) Subject Classifications: 35L15, 35L67, 35L70, 35B40

Key Words: partial differential equations, hyperbolic equations, wave equations, second order nonlinear equations, shocks and singularities

Work Unit Number 1 (Applied Analysis)

This article represents work performed at the Courant Institute of Mathematical Sciences and supported by the National Science Foundation Grant No. MCS-79-00812 and the Office of Naval Research Grant No. N00014-76-C-0439. An outline is to appear in the Proceedings of the National Academy of Sciences, June 1982. This report was prepared at the Mathematics Research Center, sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

LOWER BOUNDS FOR THE LIFE SPAN OF SOLUTIONS OF NONLINEAR WAVE EQUATIONS IN THREE DIMENSIONS*

Fritz John

This paper deals with existence of solutions $u(x_1,x_2,x_3) = u(x,t)$ of a nonlinear wave equation of the form

$$u_{tt} - \sum_{i,k=1}^{3} a_{ik}(u')u_{x_i x_k} = 0 \tag{1a}$$

for large times t. Here u' stands for the gradient vector

$$u' = (u_{x_1},u_{x_2},u_{x_3},u_t) = (D_1u,D_2u,D_3u,D_4u) = Du \tag{1b}$$

We assume that the $a_{ik}(U)$ are in C^∞ in a closed ball $|U| \le \delta$ in R^4, and that

$$a_{ik}(0) = \delta_{ik} , \tag{1c}$$

so that (1a) goes over into the classical linear wave equation

$$\Box u = u_{tt} - \Delta u = 0 \tag{1d}$$

for "infinitesimal" u. The solution u of (1a) is to be found from initial conditions for $t = 0$. Here we use initial data of the form

$$u(x,0) = \varepsilon f(x), \quad u_t(x,o) = \varepsilon g(x) \quad \text{for} \quad x \in R^3 \tag{1e}$$

where f,g are fixed functions in $C_0^\infty(R^3)$ and ε is a parameter that serves to measure the amplitude of the initial values.

For a given choice of functions $f(x),g(x),a_{ik}(U)$ we define the life span $T = T(\varepsilon)$ as the supremum of all s such that a C^∞-solution of (1a,e) with $|u'| < \delta$ exists for $x \in R^3$ and $0 \le t < s$. One knows that $T(\varepsilon) > 0$ for sufficiently small $|\varepsilon|$; ("local" solutions of the initial value problem exist). Existence of "global" solutions would correspond to $T(\varepsilon) = \infty$. One knows also (see [4]) that $T(\varepsilon) < \infty$ at least in some cases;

*This is a continuation of the author's paper DELAYED SINGULARITY FORMATION OF SOLUTIONS OF NONLINEAR WAVE EQUATIONS IN HIGHER DIMENSIONS, Comm. Pure Appl. Math. 29, (1976), 649-682, referred to as (*) in the sequel.

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(the local solution "blows up" in finite time). Thus for the equation*

$$u_{tt} - \frac{\Delta u}{1 - 2u_t} = 0 \tag{2a}$$

we have $T(\epsilon) < \infty$ for all sufficiently small positive ϵ, when

$$\int_{R^3} g(x)dx > 0 \tag{2b}$$

Realistic bounds for $T(\epsilon)$ are difficult to obtain. In the example (2a,b) one can show that

$$T(\epsilon) < A \exp(B\epsilon^{-2}) \tag{2c}$$

with certain constants A,B. In the present paper we show that $T(\epsilon)$ increases with diminishing ϵ faster than any reciprocal, power of ϵ:

THEOREM

<u>For any real positive</u> N

$$\lim_{\epsilon\to 0} |\epsilon|^N T(\epsilon) = \infty \tag{3}$$

<u>Remarks</u>

(a) Statement (3) could be wrong for <u>plane wave</u> solutions of (1a,e). Indeed it does not hold for such solutions for $N > 1$, if equation (1a) is genuinely nonlinear. But plane wave solutions are excluded by our assumption that f and g have compact support.

(b) Relation (3) had been proved in (*) for N restricted to the interval $0 < N < 4$. The proof given here for general N closely follows the ideas developed in (*). The extension to $N > 4$ requires some not so obvious additional estimates, contained in the MAIN LEMMA below.

(c) The arguments leading to (3) would also permit to derive more specific lower bounds for $T(\epsilon)$ for fixed ϵ for specific f,g,a_{ik}. These bounds would depend on assumptions on the <u>growth</u> of the derivatives of those functions <u>with order.</u> The methods used here do not yield (3) for general N, when only a <u>finite</u> number of derivatives of coefficients and data are available.

This contrasts with the situation in more than 5 space dimensions, where $T(\epsilon) = \infty$ for all sufficiently small ϵ, as shown by Klainerman [1]. See also [2], [3].

The proof of the THEOREM is broken up into a sequence of lemmas. Roughly one argues as follows. First a combination of a priori L_2-estimates ("energy" estimates) and Sobolev inequalities leads to a <u>local</u> existence proof. To get existence for really large times t one has to establish pointwise <u>decay</u> of u' with time. Decay cannot be inferred easily from L_2-estimates. It is established here by approximating u' in the L_2-norm by vectors $\tilde{u}'$ that can be shown to decay. We select here for $\tilde{u}$ the partial sums of the formal power series of u in terms of ε:

$$u \approx \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots \tag{4}$$

The $u_k(x,t)$ can be found explicitly by quadratures from linear recursion formulae. The THEOREM follows if one can prove that the derivatives of the u_k decay at least like $t^{-1}(\log t)^{2k-2}$ for large t. (The solutions of the <u>linear</u> wave equation (1d) decay like t^{-1} for initial data of compact support). In (*) this is shown for $k = 1,2,3$. The crude technique used there, only estimating absolute values, does not work for $k > 3$. One has to rely instead on <u>cancellation</u> of the worst contributions (exploited here by integration by parts). This comes about because the u_k satisfy certain <u>radiation conditions</u>, making them behave asymptotically like outgoing spherical waves. One also has to make use of the fact that the derivatives of the u_k decay more strongly, like $t^{-2}(\log t)^{2k-2}$, except for small $|x|/t$ or small $1 - |x|/t$. This is the essential content of the <u>Main Lemma</u>.

<u>Notation and assumptions.</u>

For a vector $U \in \mathbb{R}^4$ we define $|U|$ as its euclidean length.* For $U = U(x,t)$ with $x \in \mathbb{R}^3$ and a non-negative integer n we set

$$|U(x,t)|_n = \sqrt{\sum_{|\alpha| \le n} |D^\alpha U(x,t)|^2} \tag{5a}$$

and introduce for fixed t the two norms

$$\{U(t)\}_n = \sup_{x \in \mathbb{R}^3} |U(x,t)|_n \tag{5b}$$

*This differs slightly from the definition of $|U|$ in (*), (37).

$$\|U(t)\|_n = \sqrt{\int_{R^3} (|U(x,t)|_n)^2 dx} \tag{5c}$$

In what follows we shall use almost exclusively $n = 3$ in (5b) and $n = 5$ in (5c). By <u>Sobolev's inequality</u> there exists universal C such that

$$\{U(t)\}_3 \le C\|U(t)\|_5 \tag{5d}$$

We assume that the $a_{ik}(U)$ are of class C^∞ in the closed ball

$$|U| \le \delta \tag{6a}$$

and satisfy

$$a_{ik}(U) = a_{ik}(U) \tag{6b}$$

$$a_{ik}(0) = \delta_{ik} \tag{6c}$$

Without restriction generality we can assume here that δ is so small that

$$0 < \delta < 1 \tag{6d}$$

and that

$$\frac{1}{2}\,|V|^2 \le V_4^2 + \sum_{i,k=1}^{3} a_{ik}(U)V_iV_k \le 2|V|^2 \tag{6e}$$

for all U satisfying (6a) and all $V = (V_1,V_2,V_3,V_4) \in R^4$. With a given vectorfield $U = U(x,t)$ we associate the <u>linear</u> differential operator

$$P(U) = D_4^2 - \sum_{i,k=1}^{3} a_{ik}(U)D_iD_k \tag{7a}$$

The result of applying $P(U)$ to a scalar function $u(x,t)$ will be written $P(U)[u]$. In this notation (1a) becomes $P(u')[u] = 0$.

LEMMA I. (a priori estimate).

Let for a certain $T > 0$ the function $u(x,t)$ be a C^∞-solution of

$$P(u')[u] = 0 \quad \text{for} \quad x \in R^3, \quad 0 \le t \le T \tag{8a}$$

for which

$$|u'(x,t)| \le \delta \tag{8b}$$

Set

$$\lambda(t) = \max_{1\le k\le 5} (\{u'(t)\}_3)^k \tag{8c}$$

There exists a constant C (depending only on the suprema of the $a_{ik}(U)$ and their derivatives of orders ≤ 5 for $|U| \leq \delta$), such that*

$$\|u'(t)\|_5 \leq 2\|u'(0)\|_5 \exp\left(C \int_0^t \lambda(s)ds\right) \tag{8d}$$

Proof. See (*), p. 659 for $N = 5$, $N' = 3$. The extra factor 2 in (8d) is due to the difference in the definitions of $|U|$, and the use of (6e). ∎

We assign initial data

$$u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x) \tag{9a}$$

to the solution $u(x,t)$ of (8a), where $f,g \in C_0^\infty(\mathbb{R}^3)$. Without restriction of generality we can assume that

$$f(x) = g(x) = 0 \quad \text{for} \quad |x| > 1 . \tag{9b}$$

Higher derivatives of $u(x,t)$ for $t = 0$ can be computed from f,g with the help of the differential equation (8a). In particular there exists a constant c (depending on the f,g,a_{ik}) such that

$$\|u'(0)\|_5 \leq c\varepsilon \quad \text{for} \quad |\varepsilon| \leq 1 \tag{9c}$$

LEMMA II. (Local existence).

Let for given f,g,a_{ik}

$$|\varepsilon| < \min\left(1, \frac{\delta}{4cC}\right) \tag{10a}$$

We can find a value $T_0 = T_0(\varepsilon) > 0$ and a solution $u \in C^\infty$ of

$$P(u')[u] = 0 \quad \text{for} \quad x \in \mathbb{R}^3, \ 0 \leq t \leq T_0 \tag{10b}$$

$$u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x) \quad \text{for} \quad x \in \mathbb{R}^3 \tag{10c}$$

for which

$$\int_0^{T_0} \{u'(s)\}_3 ds = \frac{1}{C} \log 2 \tag{10d}$$

$$\|u'(t)\|_5 \leq 4\|u'(0)\|_5 \leq 4c|\varepsilon| < \frac{\delta}{C} \quad \text{for} \quad 0 \leq t \leq T_0 \tag{10e}$$

*We can use the same C in (5d) and (8d).

$$|u'(x,t)| < \delta < 1 \quad \text{for} \quad x \in R^3, \quad 0 \leq t \leq T_0 \tag{10f}$$

$$u(x,t) = 0 \quad \text{for} \quad |x| > t+1, \quad 0 \leq t \leq T_0 \tag{10g}$$

<u>Proof.</u> By (5b), (5d), (9c), (10a)

$$|u'(x,0)| \leq \{u'(0)\}_3 \leq C\|u'(0)\|_5 \leq cC|\varepsilon| \leq \frac{\delta}{4} < \delta . \tag{11a}$$

Let $T = T(\varepsilon)$ be the life span, as defined on p. 1. Here $T(\varepsilon) > 0$ because of (11a).

Then either

$$\lim_{t \to T} \int_0^t \lambda(s)ds = \infty \tag{11b}$$

or

$$\sup|u'(x,t)| = \delta \quad \text{for} \quad x \in R^3, \quad 0 \leq t < T \tag{11c}$$

(See (*), pp. 660-661). We define $T_0 = T_0(\varepsilon)$ by

$$\int_0^{T_0} \lambda(s)ds = \frac{\log 2}{C} \quad \text{when} \quad \int_0^T \lambda(s)ds > \frac{\log 2}{C} \tag{11d}$$

$$T_0 = T \qquad\qquad \text{when} \quad \int_0^T \lambda(s)ds \leq \frac{\log 2}{C} \tag{11e}$$

In either case

$$\int_0^{T_0} \lambda(s)ds \leq \frac{\log 2}{C} \tag{11f}$$

This implies by (8d), (9c), (10a), (5d) that for $0 \leq t < T_0$

$$\|u'(t)\|_5 \leq 4\|u'(0)\|_5 \leq 4c|\varepsilon| \tag{11f*}$$

$$\{u'(t)\}_3 \leq 4cC|\varepsilon| < \delta < 1 \tag{11g}$$

But then also by (10a)

$$|u'(x,t)| \leq 4cC|\varepsilon| < \delta \quad \text{for} \quad x \in R^3, \quad 0 \leq t < T_0 \tag{11h}$$

This is incompatible with

$$\int_0^T \lambda(s)ds \leq \frac{\log 2}{C} < \infty$$

which would imply (11c) with $T = T_0$. Hence

$$\int_0^{T_0} \lambda(s)ds = \frac{\log 2}{C}, \quad T_0 < T \tag{11i}$$

It follows from (11g) and the definition (8c) of $\lambda(t)$ that

$$\lambda(t) = \{u'(t)\}_3 \quad \text{for} \quad 0 \leq t \leq T_0 . \tag{11j}$$

Thus (11i) implies (10d). Moreover (11f*), (11h) yield (10e), (10f). Finally (10g) is a consequence of assumption (9b); the effect of zero initial data is the same as for the linear wave equation; (see [4], p. 49). ∎

Relation (10d) permits to derive lower bounds for T_0 and hence for T from upper bounds for $\{u'(t)\}_3$. Trivially one has from (11g), (10d)

$$4cC^2T|\varepsilon| \geq 4cC^2T_0|\varepsilon| \geq \log 2 \tag{12}$$

Not much more can be extracted from an upper bound for $\{u'(t)\}_3$ as long as this bound does not show decay in t. Bounds showing decay cannot be obtained from bounds for $\|u'(t)\|_5$, which is not likely to decay; (it does not in the linear case). One way to find better estimates for $\{u'(t)\}_3$ is to compare u with an "approximation" $\tilde{u}$, for which $\{\tilde{u}'\}_3$ shows the appropriate decay.

LEMMA III.

Let ε, u, T_0 be as in Lemma II. Let $\tilde{u}$ be an "approximation" of u, for which

$$P(u')[\tilde{u}] = w(x,t) \quad \text{for} \quad x \in R^3, \quad 0 \leq t < T_0 \tag{12a}$$

$$\tilde{u}(x,0) = u(x,0); \quad \tilde{u}_t(x,0) = u_t(x,0) \tag{12b}$$

There exists a constant C^* depending on the a_{ik}, but not on ε and $\tilde{u}$ such that

$$\|(u' - \tilde{u}')(t)\|_5 \leq 4M \int_0^{T_0} \|w(s)\|_5 ds \quad \text{for} \quad 0 \leq t < T_0 \tag{12c}$$

where

$$M = \exp\left(C^* \int_0^{T_0} \{\tilde{u}'(s)\}_5 ds\right) \tag{12d}$$

It follows from (5d) that

$$\{u'(t)\}_3 \leq \{\tilde{u}'(t)\}_3 + 4MC \int_0^{T_0} \|w(s)\|_5 ds \tag{12e}$$

Proof. See (*), pp. 662, where (12a) is written as a symmetric hyperbolic system for the vector $\tilde{u}'$. The factor 4 in (12c) arises again from the difference in the definitions of

$|U|$. The norm $\|w(s)\|_5$ for the scalar w is defined exactly as that for vectors U in (5a,c). ∎

We shall apply Lemma III to the case where $\tilde{u}$ is one of the partial sums of the formal power series expansion for u with respect to ε:

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \ldots \tag{13a}$$

To get recursion formulae for the u_k we expand the operator $P(U)$ of (7a) formally with respect to U:

$$P(U) = \Box + P_1(U) + P_2(U) + \ldots \tag{13b}$$

where $P_k(U)$ is a form of degree k in the components of U with coefficients, that are quadratic in D_1, D_2, D_3. Substituting (13a,b) into the equation $P(u')[u] = 0$ and comparing terms with the same power of ε we arrive at a sequence of equations

$$\Box u_N = Q_N \tag{13c}$$

for $N = 1,2,3,\ldots$. Here Q_N is a polynomial in the first and second derivatives of the u_k. Each term of this polynomial is (except for a constant factor) of the form

$$\prod_{i=1}^{r} (D^{\alpha_i} u_{k_i}) \tag{13d}$$

with multi-indices α_i and integers k_i, where $|\alpha_i| = 1,2$ and

$$\sum_{i=1}^{r} k_i = N; \quad k_i \geq 1; \quad r \geq 2 \tag{13e}$$

In particular

$$Q_1 = 0$$

$$Q_2 = -P_1(u_1')[u_1] \tag{13f}$$

$$Q_3 = -P_1(u_1')[u_2] - P_1(u_2')[u_1] - P_2(u_1')[u_1] \tag{13g}$$

$$Q_4 = -P_1(u_1')[u_3] - P_1(u_2')[u_2] - P_1(u_3')[u_1]$$

$$-P_2(u_1')[u_2] - 2P_2(u_1',u_2')[u_1] - P_3(u_1')[u_1] \tag{13g}$$

(with $P_2(U,V)$ denoting the polar from of $P_2(U)$). The equations (13c) combined with the initial conditions

$$u_1 = f(x), \quad D_4 u_1 = g(x) \quad \text{for} \quad t = 0 \tag{13h}$$

$$u_k = 0, \quad D_4 u_k = 0 \quad \text{for} \quad t = 0 \quad \text{when} \quad k > 1 \tag{13i}$$

recursively determine the u_N. Setting

$$\tilde{u}_N = \varepsilon u_1 + \varepsilon^2 u_2 + \dots + \varepsilon^N u_N \tag{13j}$$

we shall have to estimate $\|P(u')[\tilde{u}_N]\|_5$ and $\{\tilde{u}_N'\}_5$ and then apply Lemma III.

The required estimates for the $\tilde{u}_N$ involve the asymptotic behavior of the u_k for large t. To describe this behavior adequately we introduce the <u>radiation operators</u>

$$L_i = D_i - \sum_{k=1}^{3} x_i x_k |x|^{-2} D_k \quad \text{for} \quad i = 1,2,3 \tag{14a}$$

$$L_4 = D_4 + \sum_{k=1}^{3} x_k |x|^{-1} D_k \tag{14b}$$

Finally we denote by $\Box^{-1} w(x,t)$ the solution v of the equation

$$\Box v = w(x,t) \tag{14c}$$

with vanishing initial data:

$$v(x,0) = v_t(x,0) = 0 \tag{14d}$$

We first note the asymptotic behavior of solutions of the linear wave equation.

LEMMA IV.

Let $v(x,t)$ be the solution of

$$\Box v(x,t) = 0 \quad \text{for} \quad x \in R^3, \quad t > 0 \tag{15a}$$

$$v(x,0) = f(x), \quad v_t(x,0) = g(x) \tag{15b}$$

where $f \in C^3$, $g \in C^2$ and

$$f(x) = g(x) = 0 \quad \text{for} \quad |x| > 1 \tag{15c}$$

Then

$$v(x,t) = 0 \quad \text{for} \quad |t - |x|| > 1 \tag{15d}$$

$$v = 0\left(\frac{1}{t+2}\right); \quad D_i v = 0\left(\frac{1}{t+2}\right), \quad L_i v = 0\left(\frac{1}{(t+2)^2}\right) \tag{15e}$$

Here "0" stands for a constant depending on $\{f\}_2$ and $\{g\}_1$.

Proof. Classically

$$v = v_1(x,t) + D_4 v_2(x,t) \tag{16a}$$

where

$$v_1 = \frac{1}{4\pi t} \iint_{|y-x|=t} g(y)\,dS_y; \quad v_2 = \frac{1}{4\pi t} \iint_{|y-x|=t} f(y)\,dS_y \tag{16b}$$

(15d) is obvious from (15c) since for $|t - |x|| > 1$ the ball $|y| < 1$ and the sphere $|y - x| = t$ do not intersect. Since the area of intersection of the ball and the sphere is at most equal to $\min(4\pi, 4\pi t^2)$ it follows immediately that

$$v_1(x,t) \leq \left(\min\left(\frac{1}{t}, t\right)\right)\sup|g| = 0\left(\frac{1}{t+2}\right) \tag{16c}$$

The same argument shows that $D_i v_1 = 0(1/(t+2))$, where

$$D_i v_1 = \frac{1}{4\pi t} \iint_{|y-x|=t} D_i g(y)\,dS_y \quad \text{for } i = 1,2,3 \tag{16d}$$

$$D_4 v_1 = \frac{1}{4\pi t} \iint_{|y-x|=t} \left(\frac{dg}{dn} + \frac{g}{t}\right)dS_y \tag{16e}$$

Here

$$\frac{dg}{dn} = \sum_{k=1}^{3} \xi_k D_k g(y) \tag{16f}$$

is the normal derivative of g on the sphere $|y - x| = t$ with direction cosines

$$\xi_k = \frac{1}{t}(y_k - x_k) \tag{16g}$$

of the exterior normal.

The same rate of decay is then found for $D_4 v_2$ and for

$$D_i D_4 v_2 = D_4 \frac{1}{4\pi t} \iint_{|y-x|=t} D_i f\,dS_y \quad \text{for } i = 1,2,3$$

$$D_4^2 v_2 = \Delta v_2 = \frac{1}{4\pi t} \iint_{|y-x|=t} \Delta f\,dS_y$$

completing the proof of

$$v = 0\left(\frac{1}{t+2}\right), \quad D_i v = 0\left(\frac{1}{t+2}\right) \tag{16h}$$

One easily verifies, by transforming the surface integral (16b) for v_1 into a volume integral, that for $i = 1,2,3$

$$D_i v_1 = \frac{1}{4\pi t} \iint_{|y-x|=t} \xi_i\left(\frac{2g}{t} + \frac{dg}{dn}\right)dS_y - \frac{1}{4\pi t^2} \iint_{|y-x|=t} \left(2\xi_i g + y_i \frac{dg}{dn} - x_i \frac{dg}{dn}\right)dS_y \qquad (16i)$$

It follows that

$$L_i v_1 = \frac{1}{4\pi t^2} \iint_{|y-x|=t} \left(2\xi_i g + y_i \frac{dg}{dn}\right)dS_y$$

$$- \sum_{k=1}^{3} \frac{1}{4\pi t^2} x_i x_k |x|^{-2} \iint_{|y-x|=t} \left(2\xi_k g + y_k \frac{dg}{dn}\right)dS_y = 0((t+2)^{-2})$$

for $i = 1,2,3,$ since

$$\iint_{\substack{|y-x|=t\\|y|<1}} dS_y < 4\pi; \quad |\xi_i| < 1; \quad |y_i| < 1$$

Moreover by (16e,i), (15d)

$$L_4 v_1 = \frac{1}{4\pi t^2} \iint_{|y-x|=t} \left(1 + 2\sum_k x_k \xi_k |x|^{-1}\right) g \, dS_y$$

$$+ \frac{1}{4\pi t^2} \iint_{|y-x|=t} \left((t - |x|) + \sum_k y_k x_k |x|^{-1}\right) \frac{dg}{dn} dS_y = 0((t+2)^{-2})$$

Then also

$$L_i D_k v_2 = 0((t+2)^{-2}) \quad \text{for} \quad i = 1,2,3,4; \quad k = 1,2,3 ,$$

since $D_k v_2$ is obtained from v_1 by replacing g by $D_k f$. From the identities

$$L_i D_4 v_2 = L_4 D_i v_2 - \sum_{k=1}^{3} \left(x_i x_k |x|^{-2} L_4 + x_k |x|^{-1} L_i\right) D_k v_2$$

for $i = 1,2,3,$ and

$$L_4 D_4 v_2 = \sum_{k=1}^{3} \left(L_k + x_k |x|^{-1} L_4\right) D_k v_2$$

it then follows that

$$L_i D_4 v_2 = (0((t+2)^{-2}) \quad \text{for} \quad i = 1,2,3,4$$

This completes the proof of (15e). ∎

LEMMA V. (The MAIN LEMMA).

Let $w(x,t) \in C^2$ for $x \in R^3$, $t > 0$, and let

$$w(x,t) = 0 \quad \text{for} \quad |x| > t + 1 \tag{17a}$$

$$|w|, |D_i w| < M \frac{\log^k(t+2)}{(|x|+2)^2(t-|x|+2)^2} \tag{17b}$$

$$|L_i w| < M \frac{\log^k(t+2)}{(|x|+2)^2(t-|x|+2)(t+2)} \tag{17c}$$

for $i = 1,2,3,4$ with a certain k. Then

$$u = \Box^{-1} w \tag{17d}$$

satisfies

$$u(x,t) = 0 \quad \text{for} \quad |x| > t + 1 \tag{17e}$$

$$|D_i u| < AM \frac{\log^{k+2}(t+2)}{(|x|+2)(t-|x|+2)} \tag{17f}$$

$$|L_i u| < AM \frac{\log^{k+2}(t+2)}{(|x|+2)(t+2)} \tag{17g}$$

for $i = 1,2,3,4$, where A is a universal constant.

<u>Proof.</u> We postpone the lengthy proof of the MAIN LEMMA which only deals with a property of the operator $\Box$, to the Appendix, in order not to interrupt the arguments leading to the proof of the THEOREM. ∎

LEMMA VI.

Let the u_N be defined recursively for $N = 1,2,3,\ldots$ by the differential equations (13c) with initial conditions (13h,i) satisfying (9b). Then

$$u_N(x,t) = 0 \quad \text{for} \quad |x| > t + 1 \tag{18a}$$

$$D^\alpha u_N = O\left(\frac{\log^{2N-2}(t+2)}{(|x|+2)(t-|x|+2)}\right) \quad \text{for} \quad |\alpha| > 1 \tag{18b}$$

$$L_i D^\alpha u_N = O\left(\frac{\log^{2N-2}(t+2)}{(|x|+2)(t+2)}\right) \quad \text{for} \quad |\alpha| > 0 \tag{18c}$$

Here "0" stands for a constant depending on f, g, a_{ik}, N, α. Relations (18a,b) imply that

$$(u_N'(t))_k = 0\left(\frac{\log^{2N-2}(t+2)}{t+2}\right) \tag{18d}$$

$$\|u_N'(t)\|_k = 0(\log^{2N-2}(t+2)) \tag{18e}$$

for all $k \geq 0$.

<u>Proof.</u> We use induction over N. Since $\Box D^\alpha u_1 = 0$ and the initial data of $D^\alpha u_1$ have their support in the ball $|x| \leq 1$, we find from Lemma IV that (18a,b,c) holds for $N = 1$. Let (18a,b,c) hold, when N is replaced by a smaller number. Then $\Box D^\alpha u_N = D^\alpha \Box u_N$ for $|\alpha| \geq 0$ is a linear combination of terms of the form

$$\prod_{i=1}^{r} (D^{\alpha_i} u_{k_i}) \tag{19a}$$

where

$$\sum_{i=1}^{r} k_i = N; \quad k_i \geq 1; \quad r \geq 2; \quad |\alpha_i| \geq 1 \tag{19b}$$

(see (13d)). Since here $k_i \leq N - 1$ by (19b), the term (19a) can be estimated by induction assumption by

$$(|x| + 2)^{-r}(t - |x| + 2)^{-r}\log^{\mu}(t + 2)$$

with

$$\mu = (2k_1 - 2) + (2k_2 - 2) + \dots + (2k_r - 2) = 2N - 2r$$

Since $r \geq 2$ it follows that

$$\Box D^\alpha u_N = 0((|x| + 2)^{-2}(t - |x| + 2)^{-2}\log^{2N-4}(t + 2))$$

Similarly for $i = 1,2,3,4$

$$L_i \Box D^\alpha u_N = L_i D^\alpha Q_N$$

is a linear combination of terms

$$(L_i D^{\alpha_1} u_{k_1})(D^{\alpha_2} u_{k_2}) \dots (D^{\alpha_r} u_{k_r})$$

satisfying (19b). Using the induction assumption we can estimate such a term by

$$(|x| + 2)^{-r}(t + 2)^{-1}(t - |x| + 2)^{1-r}\log^{\mu}(t + 2)$$

and find that

$$L_i \Box D^\alpha u_N = O((|x| + 2)^{-2}(t - |x| + 2)^{-1}(t + 2)^{-1}\log^{2N-4}(t + 2))$$

Now

$$D^\alpha u_N = v_N + \Box^{-1} D^\alpha u_N$$

where v_N is a solution of $v_N = 0$ with the same initial values as $D^\alpha u_N$. Since all derivatives of u_N (including t-derivatives!) vanish for $|x| > 1$, it follows from LEMMAS IV and V that u_N satisfies (18a,b,c). ∎

We define $\tilde{u}_N$ by (13j). By (18d) there exists a constant Γ_N depending on f, g, a_{ik} such that

$$\{\tilde{u}_N'(t)\}_5 < \Gamma_N|\varepsilon| \frac{\log^{2N-2}(t + 2)}{t + 2} \quad \text{for} \quad |\varepsilon| < 1 \tag{20a}$$

and thus

$$\int_0^{T_0} \{\tilde{u}_N'(s)\}_5 ds < \Gamma_N|\varepsilon|\log^{2N-1}(T_0 + 2) \tag{20b}$$

We denote by E_N the set of ε for which

$$C^*\Gamma_n|\varepsilon|\log^{2n-1}(T_0(\varepsilon) + 2) < \log 2 \quad \text{for} \quad n = 1,2,\ldots,N \tag{20c}$$

$$|\varepsilon| < 1; \quad |\varepsilon| < \frac{\delta}{4cC} \tag{20d}$$

with C^* as in LEMMA III and c as in (9c). We have

$$\exp\left(C^* \int_0^{T_0} \{\tilde{u}_N'(s)\}_5\right) ds < 2 \quad \text{for} \quad n = 1,2,\ldots,N \tag{20e}$$

when $\varepsilon \in E_N$. By definition

$$E_N \subset E_k \quad \text{for} \quad k \leq N \tag{20f}$$

LEMMA VII.

Let ε, u, T_0 be as in LEMMA II. For given N there exists a constant γ (depending on f, g, a_{ik}, N) and an integer ν such that

$$\|(u' - \tilde{u}_N')(t)\|_5 \leq \gamma|\varepsilon|^{N+1}\log^\nu(t + 2) \tag{21}$$

for $0 \leq t < T_0(\varepsilon)$ and $\varepsilon \in E_N$.

<u>Proof.</u> The proof proceeds by induction over N. (It is given for $N = 1,2,3$ in (*), pp. 675-676). Using LEMMA III we have from (20e)

$$\|(u' - \tilde{u}'_N)(t)\|_5 \le 8 \int_0^{T_0} \|w_N(s)\|_5 ds \tag{22a}$$

where

$$w_N = P(u')[\tilde{u}_N] = \sum_{j=1}^{N} P(u')[\varepsilon^j u_j] \tag{22b}$$

$$w_N = w_N^* + w_N^{**} + w_N^{***} \tag{22c}$$

with

$$w_N^* = \sum_{j=1}^{N} \Big((P(u') - \Box - \sum_{k=1}^{N-j} P_k(u')\Big)[\varepsilon^j u_j] \tag{22d}$$

$$w_N^{**} = \sum_{j=1}^{N} \Big(\Box + \sum_{k=1}^{N-j} P_k(\tilde{u}'_{N-j-k+1})\Big)[\varepsilon^j u_j] \tag{22e}$$

$$w_N^{***} = \sum_{j=1}^{N} \sum_{k=1}^{N-j} \Big(P_k(u') - P_k(\tilde{u}'_{N-j-k+1})\Big)[\varepsilon^j u_j] \tag{22f}$$

(This is the analogue of the decomposition for $N = 3$ in (*), p. 676, formula (145)).

Setting

$$P(u') - \Box - \sum_{k=1}^{N-j} P_k(u') = R(u') = \sum_{i,k=1}^{3} b_{ik}(u')D_i D_k \tag{22g}$$

we see that

$$D^\alpha(R(u')[\varepsilon^j u_j]) \tag{22h}$$

is a linear combination of terms of the form

$$(\delta^\beta b_{ik}(u'))(D^{\beta_1}u)(D^{\beta_2}u) \ldots (D^{\beta_r}u)\varepsilon^j D^\gamma u_j \tag{22i}$$

where δ stands for differentiation with respect to u' and the multi-indices $\beta,\beta_1,\ldots,\beta_r,\gamma$ satisfy

$$|\beta| = r \ge 0; \quad |\beta_1| + |\beta_2| + \ldots + |\beta_r| + |\gamma| = |\alpha| + r + 2 \tag{22j}$$

$$|\beta_1| \ge 2, \quad |\beta_2| \ge 2,\ldots,|\beta_r| \ge 2, \quad |\gamma| \ge 2 \tag{22k}$$

Here $b_{ik}(u')$ is the truncated Taylor expansion of $-a_{ik}(u')$ starting with terms of order $N - j + 1$. It follows then from the boundedness of the derivatives of the $a_{ik}(U)$ for $|U| < \delta$ that

$$\delta^{\beta} b_{ik}(u') = \begin{cases} 0(|u'|^{N-j+1-r}) & \text{for} \quad r < N - j + 1 \\ 0(1) & \text{for} \quad r > N - j + 1 \end{cases}$$

By (22j,k) for $|\alpha| < 5$ at most one of the $|\beta_k|$ can exceed 4. Hence

$$(D^{\beta_1}u)(D^{\beta_2}u) \ldots (D^{\beta_r}u) = 0((|u'|_3)^{r-1}|u'|_5) \quad \text{for} \quad r > 1$$

It follows from (11g) that in all cases

$$D^{\alpha}(R(u')[\varepsilon^j u_j]) = 0(|\varepsilon|^N |u'|_5 \{u_j'\}_6)$$

Hence by (11f*), (18d)

$$\|w_N^*\|_5 = 0\Big(|\varepsilon|^{N+1} \frac{\log^{2N-2}(t+2)}{t+2}\Big) \tag{22l}$$

We turn to the estimate of w_N^{**}. By definition of the Q_N in (13c) we have the formal identities

$$0 = \sum_{N=1}^{\infty} \varepsilon^N Q_N(D^{\alpha}u_i)$$

$$= P(\varepsilon u_1' + \varepsilon^2 u_2' + \ldots)[\varepsilon u_1 + \varepsilon^2 u_2 + \ldots]$$

$$= \Box(\varepsilon u_1 + \varepsilon^2 u_2 + \ldots) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon^{k+j} P_k(u_1' + \varepsilon u_2' + \varepsilon^2 u_3' + \ldots)[u_j]$$

which yield actual finite identities when we collect the terms with equal powers of ε. Now the coeficients of ε^k with $k < N$ will not be affected if we replace $u_1' + \varepsilon u_2' + \varepsilon^2 u_3' + \ldots$ in the last expression by the finite sum

$$u_1' + \varepsilon u_2' + \varepsilon^2 u_3' + \ldots + \varepsilon^{N-k-j} u_{N-k-j+1}'$$

and restrict j to values $< N$ and k to values $< N - j$. This means that

$$w_N^{**} = \Box(\varepsilon u_1 + \varepsilon^2 u_2 + \dots + \varepsilon^N u_N)$$

$$+ \sum_{j=1}^{N} \sum_{k=1}^{N-j} \varepsilon^{j+k} P_k(u_1' + \varepsilon u_2' + \dots + \varepsilon^{N-k-j} u'_{N-k-j+1})(u_j)$$

contains no terms ε^m with $m \le N$ and is equal to a linear combination of terms of the form

$$\varepsilon^{N+1+i}(D^{\beta_1} u_{k_1})(D^{\beta_2} u_{k_2}) \dots (D^{\beta_r} u_{k_r})$$

with

$$i \ge 0; \quad r \ge 2; \quad |\beta_1| \ge 1, \dots, |\beta_r| \ge 1; \quad 2 \le r \le N .$$

The same holds for any $D^\alpha w_N^{**}$. It follows from (18d,e) that

$$\|w_N^{**}(t)\|_5 = O\Big(\varepsilon^{N+1} \sup_{i,j \le N} (\{u_i'(t)\}_5)^{N-1} \|u_j'(t)\|_6\Big)$$

$$= O\Big(\varepsilon^{N+1} \frac{\log^{4N-4}(t+2)}{t+2}\Big) \tag{22m}$$

Finally

$$P_k(u') - P_k(\tilde{u}'_{N-j-k+1})$$

is a form of degree k in $\tilde{u}'_{N-j-k+1}$ and $u' - \tilde{u}'_{N-j-k+1}$, where each term contains at least one of the latter factors. Thus

$$D^\alpha((P_k(u') - P_k(\tilde{u}'_{N-j-k+1}))[\varepsilon^j u_j])$$

is a linear combination of terms of the form

$$\varepsilon^j (D^{\alpha_1} \tilde{u}_{N-j-k+1}) \dots (D^{\alpha_r} \tilde{u}_{N-j-k+1})(D^{\beta_1}(u - \tilde{u}_{N-j-k+1})) \dots (D^{\beta_s}(u - \tilde{u}_{N-j-k+1}))(D^\gamma u_j)$$

with

$$|\alpha_i| \ge 1; \quad |\beta_k| \ge 1; \quad |\gamma| \ge 2; \quad r + s = k; \quad |s| \ge 1; \quad N - j - k + 1 \le N - 1$$

$$|\alpha_1| + \dots + |\alpha|_r + |\beta_1| + \dots + |\beta_s| + |\gamma| = |\alpha| + 2 + r + s$$

For $|\alpha| \leq 5$ it follows that at most one of the $|\beta_k|$ can exceed 3 and that none of them exceeds 6. Using (20a), (18d) and the induction assumption we find

$$\|(P_k(u') - P_k(\tilde{u}'_{N-j-k+1}))[\varepsilon^j u_j](t)\|_5$$

$$= O(\varepsilon^{r+j}(\{u' - \tilde{u}'_{N-j-k+1})(t)\}_3)^{s-1}\|(u' - \tilde{u}'_{N-j-k+1})(t)\|_5\{u'_j(t)\}_6)$$

$$= O(\varepsilon^{r+j}(\|(u' - \tilde{u}'_{N-j-k+1})(t)\|_5)^s\{u'_j(t)\}_6)$$

$$= O(\varepsilon^{r+j+(N-j-k+2)s}(t + 2)^{-1}\log^\mu(t + 2))$$

with a certain μ. Here, since $s \geq 1$ and $r + s = k$,

$$r+j+(N-j-k+2)s = r+j+s+(N-j-k+1)s \geq r+j+s+(N-j-k+1) = N+1$$

It follows that

$$\|w_N^{***}(t)\|_5 = O\left(\varepsilon^{N+1} \frac{\log^\mu(t + 2)}{t + 2}\right) .$$

Altogether then

$$\|w_N(t)\|_5 = O\left(\varepsilon^{N+1} \frac{\log^\mu(t + 2)}{t + 2}\right)$$

with a certain μ. Hence by (22a)

$$\|u' - \tilde{u}'_N)(t)\|_5 = O(\varepsilon^{N+1}\log^{\mu+1}(t + 2)) .$$

This completes the proof of (21) by induction over N, provided we still verify the case $N = 1$. In that case w_1^{***} and $w_1^{**} = \Box\varepsilon u_1$ vanish. By (221)

$$\|w_1^*\|_5 = O(\varepsilon^2(t + 2)^{-1}) ,$$

and (22a) furnishes the desired relation

$$\|u' - \tilde{u}'_1)(t)\|_5 = O(\varepsilon^2\log(t + 2)) \qquad \blacksquare$$

LEMMA VIII

$$\lim_{\varepsilon \to 0} |\varepsilon|^M T_0(\varepsilon) = \infty \tag{23}$$

(This implies the THEOREM, since $T(\varepsilon) \geq T_0(\varepsilon)$).

Proof. Assume that (23) does not hold. There exists then a constant K and a sequence of ϵ_j tending to zero for which

$$|\epsilon_j|^N T_0(\epsilon_j) < K .$$

Then $\epsilon = \epsilon_j$ will satisfy (20c,d) for all sufficiently large j, and thus $\epsilon_j \in E_N$. By (21), (5d), (20a)

$$\{u'(t)\}_3 < \{\tilde{u}_N'(t)\}_3 + \gamma C|\epsilon|^{N+1}\log^\nu(t + 2)$$

$$< \Gamma_N|\epsilon| \frac{\log^{2N-2}(t + 2)}{t + 2} + \gamma C|\epsilon|^{N+1}\log^\nu(t + 2)$$

for $\epsilon = \epsilon_j$ with j sufficiently large. It follows from (10d) that

$$\frac{\log 2}{C} = \int_0^{T_0} \{u'(s)\}_3 ds$$

$$< \Gamma_N|\epsilon_j|\log^{2N-1}(T_0(\epsilon_j) + 2) + \gamma C|\epsilon_j|^{N+1}T_0(\epsilon_j)\log^\nu(T_0(\epsilon_j) + 2)$$

$$< \Gamma_N|\epsilon_j|\log^{2N-1}(T_0(\epsilon_j) + 2) + \gamma CK|\epsilon_j|\log^\nu(T_0(\epsilon_j) + 2) \tag{24}$$

Since here

$$\log(T_0(\epsilon_j) + 2) < \log(K|\epsilon_j|^{-N} + 2); \qquad \lim_{j\to\infty} \epsilon_j = 0$$

(24) leads to a contradiction for $j \to \infty$. Thus (23) holds. ∎

APPENDIX

THE MAIN LEMMA

Let $w(x,t) \in C^2$ for $x \in R^3$, $t > 0$, and let

$$w(x,t) = 0 \quad \text{for} \quad |x| > t + 1 \tag{25a}$$

$$|w|, |D_i w| < \frac{M \log^k(t + 2)}{(|x| + 2)^2 (t - |x| + 2)^2} \quad \text{for} \quad |x| < t + 1 \tag{25b}$$

$$|L_i w| < \frac{M \log^k(t + 2)}{(|x| + 2)^2 (t - |x| + 2)(t + 2)} \quad \text{for} \quad |x| < t + 1 \tag{25c}$$

with a certain $k > 0$ and L_i defined by (14a,b) for $i = 1,2,3,4$. Then

$$u = \Box^{-1} w \tag{25d}$$

satisfies

$$u(x,t) = 0 \quad \text{for} \quad |x| > t + 1 \tag{25e}$$

$$|D_i u| < \frac{AM \log^{k+2}(t + 2)}{(|x| + 2)(t - |x| + 2)} \quad \text{for} \quad |x| < t + 1 \tag{25f}$$

$$|L_i u| < \frac{AM \log^{k+2}(t+2)}{(|x| + 2)(t + 2)} \quad \text{for} \quad |x| < t + 1 \tag{25g}$$

for $i = 1,2,3,4$ with a universal constant A.

__Proof.__

By Duhamel's principle

$$u(x,t) = \Box^{-1} w(x,t) = \int_0^t \omega(x,t,s)\,ds \tag{26a}$$

where $\omega(x,t,s)$ for $x \in R^3$, $t > s$ is the solution of

$$\Box\omega(x,t,s) = 0 \quad \text{for} \quad x \in R^3, \quad 0 < s < t \tag{26b}$$

$$\omega(x,t,s) = 0, \quad \omega_t(x,t,s) = w(x,s) \quad \text{for} \quad t = s \tag{26c}$$

For ω we have (see (16b)) the integral representation

$$\omega(x,t,s) = \frac{1}{4\pi(t - s)} \int_{|y-x|=t-s} w(y,s)\,dS_y \tag{26d}$$

(25e) is an immediate consequence of (25a), (26d) since $w(y,s) = 0$ for $|y| > s + 1$, and

$$|y| > |x| - t + s > s + 1 .$$

for $|y - x| = t - s, \quad |x| > t + 1.$

Since by (25a,b)

$$w(x,t) = O\left(M \frac{\log^k(t+2)}{(t+2)^2}\right) = O(M) \tag{26e}$$

we have the trivial estimates

$$\omega = O(M(t - s)); \quad u = \Box^{-1} w = O(Mt^2) \tag{26f}$$

We observe that $D_i u$ is the solution of

$$\Box D_i u = D_i w$$

with initial values

$$D_i u = 0, \quad D_4 D_i u = \delta_{i4} w(x,0) \quad \text{for} \quad t = 0 \tag{26g}$$

Thus

$$D_i u = \Box^{-1} D_i w + \delta_{i4} u^0 \tag{26g}$$

where u^0 is the solution of

$$\Box u^0 = 0 \tag{26h}$$

$$u^0 = 0, \quad u^0_t = w(x,0) \quad \text{for} \quad t = 0 \tag{26i}$$

Since also $D_i w = O(M)$, we find in analogy to (26f)

$$D_i u = O(Mt^2 + Mt) \tag{26j}$$

In particular $D_i u = O(M)$, if we prescribe a numerical upper bound for t, say $t < 10$. It follows that (25f,g) are satisfied for $t < 10$ since here $|x|$ is restricted to values less than $t + 1$. In what follows we can assume that

$$t > 10 \tag{27}$$

<u>Modified integral representation for</u> u.

In (26e) introduce spherical coordinates θ, ϕ on the sphere $|y - x| = t - s$, with the polar axis pointing in the direction from x to 0, with θ = polar distance, ϕ = latitude. Then (see Figure 1)

$$\omega(x,t,s) = \frac{t-s}{4\pi} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} w d\phi \tag{28a}$$

Set

$$r = |x|; \quad p = s - |y|; \quad \theta = \measuredangle(0xy); \quad \psi = \measuredangle(x0y); \quad q = \cos\psi \tag{28b}$$

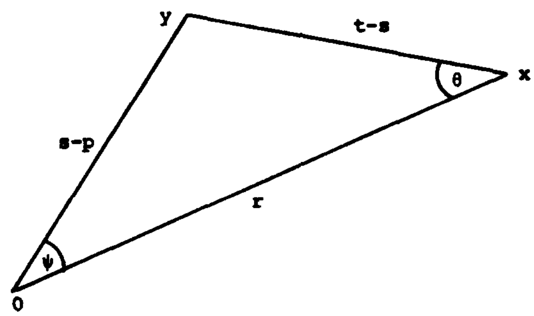

Figure 1

The ϕ-integration corresponds to y varying over the circle of intersection of the cone $y\cdot x = q|y||x|$ with the sphere $|y| = s - p$. Denote generally by j the average of w on that circle:

$$j(x,p,q,s) = \frac{1}{2\pi} \int_{\substack{y\cdot x = q|y||x| \\ |y| = s-p}} w(y,s)d\phi \tag{28c}$$

for $x \neq 0$, $|q| < 1$, $p < s$. Here ϕ is an angular measure on the circle (ϕ = arc length divided by radius). Then

$$\omega(x,t,s) = \frac{t-s}{2}\int_0^\pi j(x,p,q,s)\sin\theta d\theta \tag{28d}$$

where p and q are the functions of θ defined by

$$p = s - \sqrt{(t-s)^2 + r^2 - 2(t-s)r\cos\theta} \tag{28e}$$

$$q = Q(p,r,s,t) = \frac{(s-p)^2 + r^2 - (t-s)^2}{2r(s-p)} \tag{28f}$$

Introducing p instead of θ as variable of integration in (28d) results in the expression

$$\omega(x,t,s) = \int_A^B \frac{s-p}{2r} k(x,p,s,t)dp \tag{28g}$$

where we define

$$k(x,p,s,t) = j(x,p,Q(p,|x|,s,t),s) \tag{28h}$$

$$B = s - |s - t + r|; \quad A = s - |t + r - s| = 2s - t - r \tag{28i}$$

We introduce new independent variables in $w(y,s)$ which are better suited for describing <u>outgoing waves.</u> For that purpose we associate with $w(y,s)$ the function v defined by

$$v(y,p,s) = w((s-p)\frac{y}{|y|}, s) \quad \text{for} \quad y \neq 0, \quad p < s, \quad 0 < s \tag{29a}$$

which is homogeneous of degree 0 in y. We have conversely

$$w(y,s) = v(y,s-|y|,s) \quad \text{for} \quad y \neq 0, \quad s > 0 \tag{29b}$$

Substituting this expression for $w(y,s)$ into (28c) and replacing y by $(s-p)y$ yields

$$j(x,p,q,s) = \frac{1}{2\pi}\int_{\substack{y\cdot x = q|x| \\ |y| = 1}} v(y,p,s)d\phi \tag{29c}$$

We observe that by assumption (25a)

$$v(y,p,s) = 0 \quad \text{for} \quad p < -1 \tag{29d}$$

and then also by (29c), (28h)

$$j(x,p,q,s) = 0, \quad k(x,p,s,t) = 0 \quad \text{for} \quad p < -1 \tag{29e}$$

It follows from (28g,i) that

$$\omega(x,t,s) = \int_{-1}^{B} \frac{s - p}{2r} k(x,p,s,t)dp \quad \text{for} \quad s < \frac{t + r - 1}{2} \tag{29f}$$

$$\omega(x,t,s) = 0 \quad \text{for} \quad s < \frac{t - r - 1}{2} \tag{29g}$$

<u>Derivatives of u. Direct estimates.</u>

Let $i = 1,2,3,4$ be fixed. Set

$$W(x,t) = D_i w(x,t) \tag{30a}$$

By (26g)

$$D_i u = \Box^{-1} W + \delta_{i4} u^0$$

Here (see (16c), (26e))

$$u^0 = O\Big(\frac{M}{t + 2}\Big) = O\Big(\frac{M}{(r + 2)(t - r + 2)}\Big)$$

since $u^0(x,t) \neq 0$ only for $t - 1 < r < t + 1$. Thus for the proof of (25f) we only have to show that

$$\Box^{-1} W = O\Big(M \frac{\log^{k+2}(t + 2)}{(r + 2)(t - r + 2)}\Big) \tag{30b}$$

We represent $\Box^{-1}W$ in complete analogy to $u = \Box^{-1}w$, introducing

$$V(y,p,s) = W((s - p)\frac{y}{|y|}, s) \tag{30c}$$

$$J(x,p,q,s) = \frac{1}{2\pi} \int_{\substack{y \cdot x = q|x| \\ |y| = 1}} V(y,p,s)d\phi \tag{30d}$$

$$K(x,p,s,t) = J(x,p,Q(p,|x|,s,t),s) \tag{30e}$$

$$\Omega(x,t,s) = \int_{A}^{B} \frac{s - p}{2r} K(x,p,s,t)dp \tag{30f}$$

with A,B,Q as in (28f,i), so that

$$\Box^{-1}W(x,t) = \int_{0}^{t} \Omega(x,t,s)ds \tag{30g}$$

For V we find from assumptions (25a,b) that

$$V(y,p,s) = 0 \quad \text{for} \quad p < 1$$

$$V(y,p,s) = O\left(\frac{M \log^k(s+2)}{(s-p+2)^2(p+2)^2}\right) \tag{30h}$$

which implies that

$$J(x,p,q,s),\ K(x,p,s,t) = 0 \quad \text{for} \quad p < -1 \tag{30i}$$

$$J(x,p,q,s),\ K(x,p,s,t) = O\left(\frac{M \log^k(s+2)}{(s-p+2)^2(p+2)^2}\right) \tag{30j}$$

We can verify (30b) immediately for bounded $t - |x|$, say for

$$t - 3 < r < t + 1 \tag{31a}$$

For by (30j) with $-1 < p < s$

$$\frac{s-p}{2r} K(x,p,s,t) = O\left(\frac{M \log^k(s+2)}{r(s-p+4)}\right)$$

while by (28i) $B \leq s - (s - t + r) = t - r \leq 3$. Thus, since here $r > \frac{1}{2}(t+2)$,

$$\Omega(x,t,s) = O\left(\int_{-1}^{3} \frac{M \log^k(s+2)}{r(s-p+4)}\, dp\right) = O\left(\frac{M \log^k(t+2)}{(t+2)(s+1)}\right)$$

$$\Box^{-1} W = O\left(\frac{M \log^{k+1}(t+2)}{t+2}\right) = O\left(\frac{M \log^{k+2}(t+2)}{(r+2)(t-r+2)}\right)$$

for r satisfying (31a). Henceforth in the proof of (30b) we can assume that

$$0 < r < t - 3; \quad t > 10 \tag{31b}$$

In this section we find the relevant estimate for $\Omega(x,t,s)$ in the case where

$$\frac{1}{2}(t + r - 1) \leq s \leq t \tag{32a}$$

without making use of the radiation conditions (25c). (32a) implies that

$$-1 \leq A = 2s - t - r \tag{32b}$$

By (30f,j)

$$\begin{aligned}\Omega(x,t,s) &= O\left(\int_A^B \frac{M \log^k(t+2)}{r(s-p+2)(p+2)^2}\, dp\right) \\ &= O\left(\frac{M \log^k(t+2)}{r(s+4)^2}\left(\frac{(B-A)(S+4)}{(A+2)(B+2)} + \log\frac{(s-A+2)(B+2)}{(s-B+2)(A+2)}\right)\right)\end{aligned} \tag{32c}$$

Here

$$-1 \leq A \leq B = s - |t - r - s| \leq s$$

$$0 \leq B - A = 2r + (t - r - s) - |t - r - s| \leq 2r$$

$$\frac{B - A}{B + 2} \leq \mathrm{Min}(1,2r) = 0\left(\frac{r}{r + 2}\right)$$

$$\log \frac{(s - A + 2)(B + 2)}{(s - B + 2)(A + 2)} = \log\left(\left(1 + \frac{B - A}{s - B + 2}\right)\left(1 + \frac{B - A}{A + 2}\right)\right)$$

$$\leq 2r\left(\frac{1}{s - B + 2} + \frac{1}{A + 2}\right)$$

$$\frac{1}{s + 4} \leq \frac{2}{t + r + 7} = 0\left(\frac{1}{t + 2}\right) = 0\left(\frac{1}{r + 2}\right)$$

Hence

$$\Omega(x,t,s) = 0\left(\frac{M \log^k(t + 2)}{(r + 2)(t + 2)}\left(\frac{1}{|t - r - s| + 2} + \frac{1}{2s - t - r + 2}\right)\right)$$

and thus

$$\int_{\frac{1}{2}(t+r-1)}^{t} \Omega(x,t,s)ds = 0\left(\frac{M \log^{k+1}(t + 2)}{(r + 2)(t + 2)}\right) \tag{32d}$$

<u>Derivative of u. Integration by parts.</u>

To complete the proof of (25f) we need in addition to (32d) that

$$I = \int_{\frac{1}{2}(t-r-1)}^{\frac{1}{2}(t+r-1)} \Omega(x,t,s)ds = 0\left(\frac{M \log^{k+2}(t + 2)}{(r + 2)(t + 2)}\right) \tag{33}$$

(Note that as in (29g)

$$\Omega(x,t,s) = 0 \quad \text{for} \quad s < \tfrac{1}{2}(t - r - 1) \tag{34a}$$

since then $B < -1$, and that

$$\tfrac{1}{2}(t - r - 1) \geq 1 \tag{34b}$$

by (31b)). The considerations leading to (32d) are insufficient for the proof of (33). The estimate (32c) still holds when A is replaced by -1, but is not good enough to yield (33), and we have to have recourse to more complicated estimates involving the raditation conditions (25c). In this section we restrict ourselves to values s with

$$\tfrac{1}{2}(t - r - 1) \leq s \leq \tfrac{1}{2}(t + r - 1) \tag{34c}$$

The relation $W = D_i w$ between w and W yields relations between v and V, namely

$$V(y,p,s) = \frac{|y|}{s-p} v_{y_i}(y,p,s) - \frac{y_i}{|y|} v_p(y,p,s) \quad \text{when } i = 1,2,3 \tag{34d}$$

$$V(y,p,s) = v_s(y,p,s) + v_p(y,p,s) \quad \text{when } i = 4 \tag{34e}$$

In order to unify the arguments we introduce

$$V^*(y,p,s) = \frac{|y|}{s-p} v_{y_i}(y,p,s); \quad v^*(y,p,s) = -\frac{y_i}{|y|} v(y,p,s) \quad \text{when } i = 1,2,3$$

$$V^*(y,p,s) = v_s(y,p,s); \quad v^*(y,p,s) = v(y,p,s) \quad \text{when } i = 4$$

so that V^* and v^* are homogeneous of degree 0 in y and

$$V(y,p,s) = V^*(y,p,s) + v_p^*(y,p,s) \tag{35a}$$

In analogy to previous notation we set again

$$J^*(x,p,q,s) = \frac{1}{2\pi} \int_{\substack{y\cdot x = q|x| \\ |y|=1}} V^*(y,p,s)d\phi \tag{35b}$$

$$j^*(x,p,q,s) = \frac{1}{2\pi} \int_{\substack{y\cdot x = q|x| \\ |y|=1}} v^*(y,p,s)d\phi \tag{35c}$$

$$K^*(x,p,s,t) = J^*(x,p,Q(p,|x|,s,t),s) \tag{35d}$$

$$k^*(x,p,s,t) = j^*(x,p,Q(p,|x|,s,t),s) \tag{35e}$$

Then by (35a), (30d), (30e)

$$J(x,p,q,s) = J^*(x,p,q,s) + j_p^*(x,p,q,s) \tag{35f}$$

$$K(x,p,s,t) = K^*(x,p,s,t) + k_p^*(x,p,s,t) \tag{35g}$$

$$- Q_p(p,r,s,t) j_q^*(x,p,Q(p,r,s,t),s) \tag{35g}$$

Consequently (see (29f)), using that $k^*(x,-1,s,t) = 0$,

$$Q(x,t,s) = \int_{-1}^{B} \frac{s-p}{2r} (K^* + k_p^* - Q_p j_q^*)dp$$

$$= \frac{s-B}{2r} k^*(x,B,s,t) + \int_{-1}^{B} \left(\frac{s-p}{2r} K^*(x,p,s,t) + \frac{1}{2r} k^*(x,p,s,t)\right)dp$$

$$- \int_{-1}^{B} \frac{s-p}{2r} Q_p(p,r,s,t) j_q^*(x,p,Q(p,r,s,t),s)dp \qquad (35h)$$

One verifies easily from (29a) and the definition of V^* that

$$V^*(y,p,s) = L_i w((s-p)\frac{y}{|y|},s) \qquad (36a)$$

for $i = 1,2,3,4,$ and that

$$v_p(y,p,s) = -\sum_{k=1}^{3} \frac{y_k}{|y|} w_{x_k}((s-p)\frac{y}{|y|},s) \qquad (36b)$$

It follows from assumptions (25b,c) that

$$V^*(y,p,s) = 0\left(\frac{M \log^k(s+2)}{(s-p+2)^2(s+2)(p+2)}\right) \qquad (36c)$$

$$v^*(y,p,s),\ v_p^*(y,p,s) = 0\left(\frac{M \log^k(s+2)}{(s-p+2)^2(p+2)^2}\right) \qquad (36d)$$

Finally we obtain from

$$v_{y_k}(y,p,s) = \frac{s-p}{|y|} L_k w((s-p)\frac{y}{|y|},s) \qquad (36e)$$

and the definition of v^* that

$$v_{y_k}^*(y,p,s) = 0\left(\frac{1}{|y|}|w| + \frac{s-p}{|y|}|L_k w|\right)$$

$$= 0\left(\frac{M \log^k(s+2)}{|y|(s-p+2)(p+2)}\left(\frac{1}{(s-p+2)(p+2)} + \frac{1}{s+2}\right)\right)$$

$$= 0\left(\frac{M \log^k(s+2)}{|y|(s-p+2)(s+2)(p+2)}\right) \qquad (36f)$$

Relations (36c,d), (35b,c,d,e) imply that

$$J^*(x,p,q,s),\ K^*(x,p,s,t) = 0\left(\frac{M \log^k(s+2)}{(s-p+2)^2(s+2)(p+2)}\right) \qquad (36g)$$

$$j^*(x,p,q,s),\ k^*(x,p,s,t) = 0\left(\frac{M \log^k(s+2)}{(s-p+2)^2(p+2)^2}\right) \tag{36h}$$

We first consider the contribution to I of the term

$$\alpha = \frac{s-B}{2r} k^*(x,B,s,t) \tag{37a}$$

in (35h). By (36h)

$$\alpha = 0\left(\frac{M \log^k(t+2)}{r(s-B+2)(B+2)^2}\right) = 0\left(\frac{M \log^k(t+2)}{r(s+4)^2}\left(\frac{1}{s-B+2} + \frac{1}{B+2} + \frac{s+4}{(B+2)^2}\right)\right) \tag{37b}$$

where

$$-1 < B = s - |s-t+r| < s;\quad \tfrac{1}{2}(t-r-1) < s < \tfrac{1}{2}(t+r-1);\quad t - r > 3$$

Thus

$$\frac{1}{s+4} = 0\left(\frac{1}{t-r+2}\right)$$

If here $r < 1$ we have from (37b)

$$\alpha = 0\left(\frac{M \log^k(t+2)}{r(t-r+2)}\right)$$

$$\int_{\frac{1}{2}(t-r-1)}^{\frac{1}{2}(t+r-1)} \alpha ds = 0\left(\frac{M \log^k(t+2)}{(t-r+2)}\right) = 0\left(\frac{M \log^k(t+2)}{(r+2)(t-r+2)}\right) \tag{37c}$$

If, on the other hand, $r > 1$ we have

$$\frac{1}{r(s+4)^2}\left(\frac{1}{s-B+2} + \frac{1}{B+2} + \frac{s+4}{(B+2)^2}\right)$$

$$= 0\left(\frac{1}{r+2}\left(\frac{1}{(s+4)^2} + \frac{1}{(s+4)(t-r+2)} + \frac{1}{(t-r+2)(2s-t+r+2)}\right)\right)$$

and hence

$$\int_{\frac{1}{2}(t-r-1)}^{\frac{1}{2}(t+r-1)} \alpha ds = 0\left(\frac{M \log^{k+1}(t+2)}{(r+2)(t-r+2)}\right) \tag{37d}$$

We next consider

$$\beta = \int_{-1}^{B}\left(\frac{s-p}{2r} K^*(x,p,s,t) + \frac{1}{2r} k^*(x,p,s,t)\right)dp \tag{38a}$$

Here by (36g,h)

$$\frac{s-p}{2r} K^* + \frac{1}{2r} k^* = O\left(\frac{M \log^k(t+2)}{r(s-p+2)(p+2)}\left(\frac{1}{s+2} + \frac{1}{(s-p+2)(p+2)}\right)\right)$$

$$= O\left(\frac{M \log^k(t+2)}{r(s-p+2)(p+2)(s+2)}\right)$$

$$= O\left(\frac{M \log^k(t+2)}{r(s+2)(s+4)}\left(\frac{1}{s-p+2} + \frac{1}{p+2}\right)\right)$$

since $p < B < s$. Hence by (34c)

$$\beta = O\left(\frac{M \log^{k+1}(t+2)}{r(s+2)(s+4)}\right)$$

$$\beta = O\left(\frac{M \log^{k+1}(t+2)}{r(t-r+2)}\right) \quad \text{for} \quad r < 1$$

$$\beta = O\left(\frac{M \log^{k+1}(t+2)}{(s+2)^2(r+2)}\right) \quad \text{for} \quad r > 1$$

In either case

$$\int_{\frac{1}{2}(t-r-1)}^{\frac{1}{2}(t+r-1)} \beta ds = O\left(\frac{M \log^{k+1}(t+2)}{(r+2)(t-r+2)}\right) \tag{38b}$$

This leaves

$$\gamma = \int_{-1}^{B} \frac{s-p}{2r} Q_p(p,r,s,t) j_q^*(x,p,Q(p,r,s,t),s) dp \tag{39a}$$

For j^* given by (35c) one easily derives the differentiation formula

$$j_q^*(x,p,q,s) = \frac{1}{2\pi r(1-q^2)} \int_{\substack{y\cdot x=qr \\ |y|=1}} \sum_{k=1}^{3} (x_k - rqy_k) v_{y_k}^*(y,p,s) d\phi$$

Here for $x\cdot y = qr$, $|y| = 1$

$$|x_k - rqy_k| \leq |x - rqy| = r\sqrt{1-q^2} \, .$$

It follows from (36f) that

$$j_q^*(x,p,q,s) = O\left(\frac{M\log^k(s+2)}{(s-p+2)(s+2)(p+2)}(1-q^2)^{-1/2}\right) \tag{39b}$$

By definition (28f) of $Q(p,r,s,t)$ we have

$$\sqrt{1-Q^2(p,r,s,t)} = \frac{R}{2r(s-p)} \tag{39c}$$

where

$$R = \sqrt{(t+r-p)(t-r-p)(t+r-2s+p(2s-t+r-p)} \tag{39d}$$

is Heron's expression for four times the area of the triangle with vertices $0,x,y$; (see Figure 1). Moreover

$$Q_p(p,r,s,t) = \frac{S}{2r(s-p)^2} \tag{39e}$$

where

$$S = -(s-p)^2 + r^2 - (t-s)^2 \tag{39f}$$

Consequently

$$Y = O\left(\int_{-1}^{B} \frac{M\log^k(t+2)}{r(s-p+2)(s+2)(p+2)}\,\frac{S}{R}\,dp\right) \tag{39g}$$

Writing

$$S = -2(s-p)^2 + (t-p)(2s-t+r-p)-r(t-r-p) \tag{40a}$$

and using that

$$0 < s\text{-}B < s\text{-}p < t\text{-}r\text{-}p; \quad 0 < B\text{-}p = 2s\text{-}t\text{+}r\text{-}p < t\text{-}r\text{-}p \quad \text{for} \quad s < t\text{-}r$$

$$0 < s\text{-}B < s\text{-}p < 2s\text{-}t\text{+}r\text{-}p; \quad 0 < B\text{-}p = t\text{-}r\text{-}p < 2s\text{-}t\text{+}r - p \quad \text{for} \quad s > t\text{-}r$$

we see that

$$S = O((t+2)(t-r-p)) \qquad \text{for} \quad s < t-r \tag{40b}$$

$$S = O((t+2)(2s-t+r-p)) \quad \text{for} \quad s > t-r \tag{40c}$$

We first take up the case

$$s > t-r \tag{40d}$$

This case only occurs when

$$r > \frac{t+1}{3} \tag{40e}$$

because of (34c). Here

$$B = t-r; \quad \frac{1}{t+r-p} = 0\left(\frac{1}{t+2}\right); \quad s-p > \tfrac{1}{2}(2s-t+r-p)$$

Hence by (40c)

$$\frac{s}{(s-p+2)(p+2)R}$$

$$= 0\left(\frac{t+2}{s+4}\left(\frac{1}{s-p+2} + \frac{1}{p+2}\right)\sqrt{\frac{2s-t+r-p}{(t+r-p)(t-r-p)(t+r-2s+p)}}\right)$$

$$= 0\left(\frac{\sqrt{t+2}}{(s+4)\sqrt{t+r-2s-1}}\left(\frac{1}{\sqrt{(t-r-p)(2s-t+r-p+4)}} + \frac{\sqrt{2s-t+r+1}}{(p+2)\sqrt{t-r-p}}\right)\right)$$

Hence by (39g), (40e)

$$Y = 0\left(\frac{M\sqrt{t+2}\,\log^{k+1}(t+2)}{r(s+2)^2\sqrt{t+r-2s-1}}\left(1 + \sqrt{\frac{2s-t+r+1}{t-r+2}}\right)\right)$$

$$= 0\left(\frac{M\log^{k+1}(t+2)}{\sqrt{(t+2)(t-r+2)}}\,\frac{\sqrt{2s-t+r+1}}{(s+2)^2\sqrt{t+r-2s-1}}\right)$$

With the substitution

$$s + 2 = \frac{(t+r+3)(t-r+3)(1+v^2)}{2(t+r+3) + 2(t-r+3)v^2}$$

we find

$$\int_{t-r}^{\frac{1}{2}(t+r-1)} Y ds = 0\,\frac{M\log^{k+1}(t+2)}{\sqrt{(t+2)(t-r+2}}\,r(t-r+3)^{-1/2}(t+r+3)^{-3/2}\int_a^{\infty}\frac{v^2 dv}{(1+v^2)^2} \tag{40f}$$

$$= 0\left(\frac{M\log^{k+1}(t+2)}{(r+2)(t-r+2)}\right)$$

with

$$a = \sqrt{\frac{(t+r+3)(t-r+1)}{(t-r+3)(3r-t-1)}}$$

We come to the contribution of the values s with [see (34c)]

$$\tfrac{1}{2}(t-r-1) < s < D = \mathrm{Min}(\tfrac{1}{2}(t+r-1), t-r) \tag{41a}$$

Here

$$S = O((t + 2)(t-r-p)), \quad B = 2s-t+r$$

$$\frac{t-r-p}{t+r-p} < \frac{t-r+1}{t+r+1} = O\left(\frac{t-r+1}{t + 2}\right)$$

$$t+r-2s+p > t+r-2s-1$$

Hence by (39g)

$$\gamma = O\left(\int_{-1}^{B} \frac{M \log^k(t + 2)}{r(s + 2)(s-p+2)(p + 2)} \sqrt{\frac{(t + 2)(t-r+1)}{(t+r-2s-1)(2s-t+r-p)}}\, dp\right)$$

$$= O\left(\frac{M \log^k(t + 2)}{r(s + 2)(s + 4)} \sqrt{\frac{(t + 2)(t-r+1)}{t+r-2s-1}}\, H\right) \tag{41b}$$

with

$$H = \int_{-1}^{2s-t+r} \left(\frac{1}{s-p+2} + \frac{1}{p + 2}\right) \sqrt{\frac{1}{2s-t+r-p}}\, dp$$

$$= \frac{1}{\sqrt{t-r-s+2}} \int_0^a \frac{d\theta}{(1 + \theta)\sqrt{\theta}} + \frac{1}{\sqrt{2s-t+r+2}} \int_0^b \frac{d\theta}{(1 - \theta)\sqrt{\theta}}$$

$$= O\left(\sqrt{\frac{a}{(a + 1)(t-r-s+2)}} + \sqrt{\frac{b}{2s-t+r+2}} \log \frac{2}{1 - b}\right) \tag{41c}$$

$$a = \frac{2s-t+r+1}{t-r-s+2}, \quad b = \frac{2s-t+r+1}{2s-t+r+2} \tag{41d}$$

Here by (41a)

$$a = \frac{2r - (t+r-1-2s)}{2 + (t-r-s)} < r; \quad \frac{a}{a + 1} < \frac{r}{r + 1}$$

$$b = \frac{2r - (t+r-1-2s)}{2r + 1 - (t+r-1-2s)} < \frac{2r}{2r + 1}$$

$$b \log \frac{2}{1 - b} < \frac{2r}{2r + 1} \log(4r + 2)$$

so that

$$\gamma = O\left(\frac{M \log^{k+1}(t + 2)}{(s + 2)(s + 4)} \sqrt{\frac{(t + 2)(t-r+1)}{r(r + 1)(t+r-2s-1)}} \left(\sqrt{\frac{1}{t-r-s+2}} + \sqrt{\frac{1}{2s-t+r+2}}\right)\right)$$

In the special case $0 < r < 1$ we have by (41a)

$$D = \tfrac{1}{2}(t+r-1); \quad s+2 > \tfrac{1}{2}(t+2); \quad t-r-s+2 > \tfrac{1}{2}(t+2); \quad 2s-t+r+2 > 1$$

Hence

$$\gamma = 0\left(\frac{M \log^{k+1}(t+2)}{(t+2)\sqrt{r(t+r-2s-1}}\right)$$

and thus

$$\int_{\frac{1}{2}(t-r-1)}^{\frac{1}{2}(t+r-1)} \gamma ds = 0\left(\frac{M \log^{k+1}(t+2)}{t+2}\right) = 0\left(\frac{M \log^{k+1}(t+2)}{(r+2)(t+2)}\right) \tag{41e}$$

In the remaining case $1 < r < t-3$ we have

$$\gamma = 0\left(\frac{M \log^{k+1}(t+2)}{(r+2)(s+2)}\sqrt{\frac{t+2}{(t-r+2)(t+r-2s-1}}\left(\sqrt{\frac{1}{t-r-s+2}} + \sqrt{\frac{1}{2s-t+r+2}}\right)\right)$$

Here

$$\int_{\frac{1}{2}(t-r-1)}^{D} \frac{1}{s+2}\sqrt{\frac{1}{(t+r-2s-1)(2s-t+r+2)}}\, ds$$

$$< \int_{\frac{1}{2}(t-r-1)}^{\frac{1}{2}(t+r-1)} \frac{1}{s+2}\sqrt{\frac{1}{(t+r-2s-1)(2s-t+r+1)}}\, ds$$

$$= 0\left(\sqrt{\frac{1}{(t+r+3)(t-r+3)}}\right) = 0\left(\sqrt{\frac{1}{(t+2)(t-r+2}}\right) \tag{44f}$$

We need a similar estimate for

$$\int_{\frac{1}{2}(t-r-1)}^{D} \frac{1}{s+2}\sqrt{\frac{1}{(t+r-2s-1)(t-r-s+2)}}\, dS = G \tag{44g}$$

Let at first $1 < r < t/2$. Then $s+2 > \frac{1}{2}(t-r+3) > (t+2)/8$, and

$$G = \frac{1}{t+2}\int_{\frac{1}{2}(t-r-1)}^{D} \sqrt{\frac{1}{(t+r-2s-1)(t-r-s+2)}}\, ds = 0\left(\frac{1}{t+2}\int_a^b \frac{d\theta}{\sqrt{\theta(1+\theta)}}\right)$$

with

$$a = 0, \quad b = \frac{2r}{t-3r+5} \quad \text{for} \quad 1 < r < \frac{t+1}{3}$$

$$a = \frac{3r-t-1}{t-3r+5}, \quad b = \frac{2r}{t-3r+5} \quad \text{for} \quad \frac{t+1}{3} < r < \frac{t+5}{3}$$

$$a = \frac{4}{3r-t-5}, \quad b = \frac{t-r+5}{3r-t-5} \quad \text{for} \quad \frac{t+5}{3} < r < \frac{t}{2}$$

In each of the three sub-cases

$$G = 0\left(\frac{\log(t+2)}{t+2}\right)$$

as is easily verified. If instead $t/2 < r < t-3$ we have

$$D = t-r; \quad t+r-2s-1 > t+r-2(t-r)-1 = 3r-t-1 > \tfrac{1}{4}(t+2)$$

$$G = 0\left(\frac{1}{\sqrt{t+2}} \int_{\frac{1}{2}(t-r-1)}^{t-r} \frac{ds}{(s+2)\sqrt{t-r-s+2}}\right) = 0\left(\frac{1}{\sqrt{(t-r+4)(t+2)}} \int_a^b \frac{d\theta}{(1-\theta)\sqrt{\theta}}\right)$$

with

$$0 < a = \frac{2}{t-r+4} < b = \frac{t-r+5}{2(t-r+4)} < \frac{4}{7}$$

Here also

$$G = 0\left(\sqrt{\frac{1}{(t+2)(t-r+2)}}\right)$$

Altogether then for $1 < r < t-3$

$$\int_{\frac{1}{2}(t-r-1)}^{D} \gamma ds = 0\left(\frac{M \log^{k+2}(t+2)}{(r+2)(t-r+2)}\right)$$

This, together with (41e), (40f), (38b), (37d), (37c), (32d) completes the proof of (25f).

<u>Proof of the radiation conditions.</u>

We notice that (25f) implies (25g) when $|x| = r < \frac{1}{2}t$. Thus in the proof of (25g) we can restrict ourselves to the case

$$\tfrac{1}{2}t < r < t+1 \tag{45a}$$

By (26d) and the assumption $w \in C^2$ we see that

$$\omega(x,ts) = \frac{t-s}{4\pi} \iint_{|\xi|=1} w(x+(t-s)\xi, s)\, dS_\xi$$

belongs to C^2 in x,t,s for $0 \leq s \leq t$. Because of (26a), (26c)

$$L_i u = \int_0^t L_i w(x,t,s)ds \tag{45b}$$

Since

$$L_i t = L_i r = \delta_{i4} \tag{45c}$$

we find from (28g), (28i) that

$$L_i u = \delta_{i4} \int_0^t \frac{s - A}{2r} k(x,A,s,t)ds - \delta_{i4} \int_0^t ds \int_A^B \frac{s - p}{2r^2} k(x,p,s,t)dp$$

$$+ \int_0^t ds \int_A^B \frac{s - p}{2r} L_i k(x,p,s,t)dp \tag{45d}$$

Take first the case $i = 1,2,3$. By (28h), (45c), (14a)

$$L_i k = \sum_{m=1}^{3} (\delta_{im} - r^{-2} x_i x_m) j_{x_m}(x,p,q,s) \tag{45e}$$

Here by (29c), (36e), (25c)

$$j_{x_m}(x,p,q,s) = \frac{1}{2\pi} \int_{\substack{y \cdot x = qr \\ |y|=1}} \left(\frac{q}{r} v_{y_m} - \sum_{n=1}^{3} r^{-2} y_m x_n v_{y_n}\right) d\phi$$

$$= 0\left(\frac{M \log^k(s + 2)}{r(s-p+2)(s + 2)(p + 2)}\right) \tag{45f}$$

It follows from (45d), (45a) that for $i = 1,2,3$

$$L_i u = 0\left(\int_0^t ds \int_A^B \frac{M \log^k(s + 2)}{r^2(s + 2)(p + 2)} dp\right)$$

$$= 0\left(\frac{M \log^{k+2}(t + 2)}{(r + 2)(t + 2)}\right) \tag{45g}$$

(More precisely the p-integration is taken over that portion of the interval (A,B) in which $p > -1$).

We turn to the case $i = 4$. By (28c,h), (25b)

$$\frac{s-p}{k}\, k(x,p,s,t) = 0\left(\frac{M \log^k(s+2)}{r(s-p+2)(p+2^2}\right)$$

$$= 0\left(\frac{M \log^k(t+2)}{r(s+4)^2}\left(\frac{1}{s-p+2} + \frac{1}{p+2} + \frac{s+4}{(p+2)^2}\right)\right)$$

It follows that

$$\int_0^t ds \int_A^B \frac{s-p}{2r^2}\, k(x,p,s,t)dp$$

$$= 0\left(\frac{M \log^{k+1}(t+2)}{r^2}\right) = 0\left(\frac{M \log^{k+1}(t+2)}{(r+2)(t+2)}\right) \tag{46a}$$

Similarly

$$\frac{s-A}{2r}\, k(x,A,s,t) = 0 \quad \text{for} \quad A = 2s-t-r < -1$$

$$\frac{s-A}{2r}\, k(x,A,s,t) = 0\left(\frac{M \log^k(t+2)}{r(t+r-s+2)(2s-t-r+2)^2}\right)$$

$$= 0\left(\frac{M \log^k(t+2)}{r(r+1)(2s-t-r+2)^2}\right) \quad \text{for} \quad A = 2s-t-r > -1$$

It follows that

$$\int_0^t \frac{s-A}{2r}\, k(x,A,s,t)ds = 0\left(\frac{M \log^k(t+2)}{(r+2)(t+2)}\right) \tag{46b}$$

This leaves the last term in (45d). By (14b), (28h)

$$L_4 k = (Q_r + Q_t)j_q(x,p,Q,s) + \sum_{m=1}^{3} r^{-1} x_m j_{x_m} \tag{47a}$$

The contribution to $L_4 u$ of the terms with j_{x_m} is again of order

$$0\left(\frac{M \log^{k+2}(t+2)}{(r+2)(t+2)}\right)$$

as follows from (45f), (45a). We estimate next

$$(Q_r + Q_t)j_q = -\frac{(2s-t+r-p)(t-r-p)}{2(s-p)r^2} j_q \tag{47b}$$

We find from (39b) (which applies just as well to j_q as to j_q^*), and (39c,d) that

$$\frac{s-p}{r}(Q_r + Q_t)j_q = O\left(\frac{M \log^k(s+2)}{r^2(s+2)(p+2)} \sqrt{\frac{(t-r-p)(2s-t+r-p)}{(t+r-p)(t+r-2s+p)}}\right)$$

Here

$$2s-t+r-p < t+r-p; \quad t-r-p < t-r+1$$

If also $s < \frac{1}{2}(t+r-1)$ we have $t+r-2+p > t+r-2s-1$, $A < -1$, and

$$\int_{-1}^{B} \frac{s-p}{r}(Q_r + Q_t)j_q dp = O\left(\frac{M \log^{k+1}(t+2)}{r^2(s+2)} \sqrt{\frac{t-r+1}{t+r-2s-1}}\right)$$

$$\int_0^{\frac{1}{2}(t+r-1)} ds \int_{-1}^{B} \frac{s-p}{r}(Q_r + Q_t)j_q dp = O\left(\frac{M \log^{k+2}(t+2)}{r^2} \sqrt{\frac{t-r+1}{t+r+3}}\right)$$

$$= O\left(\frac{M \log^{k+2}(t+2)}{(r+2)(t+2)}\right)$$

If instead $\frac{1}{2}(t+r-1) < s < t$ we have $s+2 > \frac{1}{2}(t+3)$, $A > -1$ and

$$\frac{s-p}{r}(Q_r + Q_t)j_q = O\left(\frac{M \log^k(t+2)}{r^2(t+2)(p+2)} \sqrt{\frac{t-r+1}{p-A}}\right)$$

$$\int_A^B \frac{s-p}{r}(Q_r + Q_p)j_q dp = O\left(\frac{M \log^k(t+2)}{r^2\sqrt{t+2}} \int_A^{\infty} \frac{dp}{(p+2)\sqrt{p-A}}\right)$$

$$= O\left(\frac{M \log^k(t+2)}{r^2\sqrt{(t+2)(A+2)}}\right)$$

$$\int_{\frac{1}{2}(t+r-1)}^{t} ds \int_A^B \frac{s-p}{r}(Q_r + Q_t)j_q dp = O\left(\frac{M \log^k(t+2)}{r^2}\right)$$

$$= O\left(\frac{M \log^k(t+2)}{(r+2)(t+2)}\right)$$

This completes the proof of the radiation conditions (25g) and of the MAIN LEMMA. ∎

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FJ/ed

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2393	2. GOVT ACCESSION NO. AD-A118599	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) LOWER BOUNDS FOR THE LIFE SPAN OF SOLUTIONS OF NONLINEAR WAVE EQUATIONS IN THREE DIMENSIONS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Fritz John		8. CONTRACT OR GRANT NUMBER(s) MCS-79-00812 DAAG29-80-C-0041 N00014-76-C-0439
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS (see Item 18 below)		12. REPORT DATE June 1982 13. NUMBER OF PAGES 38
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709	Office of Naval Research Arlington, VA 22217	National Science Foundation Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

partial differential equations, hyperbolic equations, wave equations, second order nonlinear equations, shocks and singularities

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper deals with strict solutions $u(x,t) = u(x_1,x_2,x_3,t)$ of an equation

$$u_{tt} - \sum_{i,k=1}^{3} a_{ik}(Du)u_{x_i x_k} = 0$$

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ABSTRACT (cont.)

where Du is the set of 4 first derivatives of u. For given initial values $u(x,0) = \varepsilon F(x)$, $u_t(x,0) = \varepsilon G(x)$ the life span $T(\varepsilon)$ is defined as the supremum of all t to which the local solution can be extended for all x. Blow-up in finite time corresponds to $T(\varepsilon) < \infty$. Examples show that this can occur for arbitrarily small ε. On the other hand $T(\varepsilon)$ must at least be very large for small ε. Assuming that $a_{ik}, F, G \in C^\infty$, that $a_{ik}(0) = \delta_{ik}$, and that F, G have compact support, it is shown that $\lim_{\varepsilon \to 0} \varepsilon^N T(\varepsilon) = \infty$ for every N. This result had been established previously only for $N < 4$.

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